MATH 210: Introduction to Analysis

Fall 2015-2016, Final, Duration: 120 min.

Exercise 1. The contraction mapping theorem. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Assume that there is $0 < k < 1$ such that $|f'(x)| \leq k < 1$ for all $x \in \mathbb{R}$. Let $x_0 \in \mathbb{R}$ be any real number and define a sequence by setting $x_{n+1} = f(x_n)$.

(a) (**5 points**) Prove that f is a contraction.
(b) (**10 points**) Recall the definition of a Cauchy sequence.
(c) (**10 points**) Prove that the sequence $\{x_n\}$ is Cauchy.
(d) (**5 points**) Deduce that the sequence $\{x_n\}$ converges to a real number x^*.
(e) (**5 points**) Prove that x^* is a fixed point of f.
(f) (**5 points**) Prove that x^* is the unique fixed point of f.

Exercise 2.

(a) (**10 points**) Let (X, d) be a metric space and let $f : X \to \mathbb{R}$ be function. State the definition of continuity of f at x_0.
(b) Let (X, d) be a metric space and let $A \subset X$ be dense in X. Assume that $f, g : X \to \mathbb{R}$ are two continuous functions such that $f(x) = g(x)$ for all $x \in A$. The goal of this question is to prove that f and g are equal. Fix $x_0 \in X \setminus A$ and $\varepsilon > 0$.
   i. (**5 points**) Find δ such that if $d(x, x_0) < \delta$ then $|f(x) - f(x_0)| < \varepsilon$ and $|g(x) - g(x_0)| < \varepsilon$.
   ii. (**3 points**) Show that for any $y \in X$, $|f(x_0) - g(x_0)| \leq |f(x_0) - f(y)| + |f(y) - g(y)| + |g(y) - g(x_0)|$.
   iii. (**10 points**) Show using the previous question that $|f(x_0) - g(x_0)| \leq 2\varepsilon$.
   iv. (**5 points**) Deduce that $f(x_0) = g(x_0)$.
   v. (**2 points**) Deduce f and g are equal.
(c) (**5 points**) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x) = 0$ for all $x \in \mathbb{Q}$. Determine f.

Exercise 3. (**20 points**) Let $n \geq 2$ be an integer. Using lower and upper sums for an appropriate partition prove that

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq \log n \leq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}.$$

Exercise 4.

(a) (**10 points**) State Rolle's Theorem.
(b) Consider the function $f(x) = \int_0^{x+x^3} (1 + sin^2 t) dt$.
   i. (**5 points**) Find a root of f.
   ii. (**10 points**) Show using Rolle's theorem that f has only one root on $\mathbb{R}$.

Exercise 5. Let $f_n : [0, 1] \to \mathbb{R}$ defined by

$$f_n(x) := \begin{cases} n^2 x(1 - nx) \text{ if } x \in [0, \frac{1}{n}] \\ \\ 0 \text{ otherwise.} \end{cases}$$

(a) (**10 points**) Study the pointwise convergence of f_n on $[0, 1]$.
(b) (**10 points**) After justifying that f_n is Riemann integrable, compute $\int_0^1 f_n(t) dt$ for all $n \geq 1$.

(c) (**10 points**) Study the uniform convergence of f_n on $[0,1]$.
(d) (**5 points**) Let $0 < a < 1$. Study the uniform convergence of f_n on $[a,1]$.

Exercise 6. The goal of this exercise is to give a new proof of the convergence of the series $\sum \frac{1}{n^\alpha}$, when $\alpha > 1$.

Let $\alpha > 1$ and consider the following function on $(0,+\infty)$ defined by

$$f(x) := -\frac{1}{(\alpha-1)x^{\alpha-1}}.$$

(a) (**5 points**) Compute f'.
(b) (**5 points**) Apply the Mean Value Theorem to f on the interval $[k,k+1]$. We denote by $x_k \in (k,k+1)$ the point obtained by this theorem.
(c) (**10 points**) Using the previous question, prove that

$$\sum_{k=1}^{n} \frac{1}{(x_k)^\alpha} = \frac{1}{\alpha-1} - \frac{1}{(\alpha-1)(n+1)^{\alpha-1}}.$$

(d) (**5 points**) Prove that $\displaystyle\sum_{k=1}^{n} \frac{1}{(k+1)^\alpha} < \sum_{k=1}^{n} \frac{1}{(x_k)^\alpha}$.
(e) (**5 points**) Deduce that the sequence $\displaystyle\sum_{k=1}^{n} \frac{1}{k^\alpha}$ is bounded.
(f) (**5 points**) Deduce that $\displaystyle\sum \frac{1}{n^\alpha}$ is convergent.

Exercise 7. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function.
(a) (**10 points**) Supose that f is twice differentiable at x_0. Prove that

$$\lim_{h\to 0} \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} = f''(x_0).$$

(b) (**5 points**) Assume that the limit of $\displaystyle\frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2}$ as $h \to 0$ exists. Disprove via a counterexample that f is twice differentiable at x_0.